MMP Learning Seminar 24/06/2022

Characterisations of CY pairs with toric models.

§ Setup - general results.

- (X,D) <u>Calabi-Yau pair</u>
  - X normal projective
  - D reduced sum of integral Weil divisors
  - $K_X + D \sim_{\mathbb{Z}} 0$

- When nothing added (X,D) log canonical (lc.)

- Will sometimes assume divisorially log terminal (dlt) or that X is terminal, $\mathbb{Q}$-factorial and (X,D) is dlt / lc $\Big\}$ (X,D) is (t, dlt) / (t, lc)

- A birational map $(X, D_X) \overset{\varphi}{\dashrightarrow} (Y, D_Y)$ is crepant if $\forall$ E geometric valuation with centre on X, Y
$$a_E(K_X + D_X) = a_E(K_Y + D_Y)$$

also called volume preserving when (X, D_X) and (Y, D_Y) are CY pairs.

- <u>Fact</u> (CK15) If (X, D_X) is a lc CY pair $\exists$ <u>crepant morphism</u>
$$\begin{array}{c} (\tilde{X}, D_{\tilde{X}}) \\ \downarrow \\ (X, D_X) \end{array}$$
where $(\tilde{X}, D_{\tilde{X}})$ is (t, dlt)

- <u>Dual complex of a CY pair</u>

(X, D_X) dlt pair, $D_X = \sum D_i$

$\mathcal{D}(X, D_X)$ = the regular cell complex obtained by attaching an $|I|-1$ dim^{al} cell $\forall$ $\phi \neq$ irreducible component of $\bigcap_{i \in I} D_i$ "encodes combinatorics of lc centres"

[De Fernex-Kollár-Xu] The PL homeomorphism type of $\mathcal{D}(X, D_X)$ is a birational invariant

$\Rightarrow$ $\dim \mathcal{D}(X, D_X)$ is a <u>crepant birational invariant</u>

Say (X, D_X) has maximal intersection if $\dim \mathcal{D}(\tilde{X}, D_{\tilde{X}}) = \dim X - 1$ where $(\tilde{X}, D_{\tilde{X}}) = a\,(t, \text{dlt})$ crepant bu of (X, D_X)

$\Leftrightarrow$ $(\tilde{X}, D_{\tilde{X}})$ has a 0-dim^{al} lc centre.

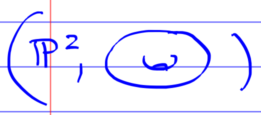
<u>Examples:</u>

(X, D_X)

X: smooth Fano

$D_X \in |-K_X|$ a smooth ac divisor

then $\mathcal{D}(X, D_X) = \bullet$

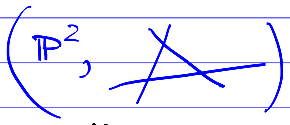


$(\mathbb{P}^n, \sum H_i)$

$\mathbb{P}^n$

H_i: coordinate hyperplanes

then $\mathcal{D}(X, D_X) = n$-simplex



[more generally
X: toric variety normal term. $\mathbb{Q}$-fact
D_X: reduced sum of T-inv^t divisors.

<u>Fact</u>: Any CY pair crepant birational to a toric pair (:= with a toric model) has maximal intersection

What is special about CY pairs with maximal intersection?

<u>Fact</u>: [Kollár–Xu 15] maximal intersection pairs have some "Fano type" properties.

(X, D_X) max intersection

$\Rightarrow$ • X is rationally connected

• $\exists$ crepant birational $(X, D_X) \overset{\varphi}{\dashrightarrow} (Z, D_Z)$ where D_Z fully supports a big semi-ample divisor.

<u>Folklore conjecture:</u>

(X, D_X) has maximal intersection $\Leftrightarrow$ (X, D_X) has a toric model.

<u>A bit of context.</u> Shokurov's conjecture

<u>Thm</u> [Brown–McKernan–Svaldi–Zong] If (X, Δ) is a lc pair and $-(K_X + \Delta)$ is nef, and if $\sum_{i=1}^{k} a_i S_i \leq \Delta$ for $S_i \geq 0$ $\mathbb{Z}$-div. $a_i \geq 0$

with $n + r - d < 1$ then $n + r - d \geq 0$ and

$n = \dim X$, $r = \dim \operatorname{Span}\{S_i\}_{1 \leq i \leq k}$, $d = \sum a_i$

$\exists D$ with (X, D) toric pair and

• $D \geq \lfloor \Delta \rfloor$

• all but $\lfloor 2c \rfloor$ components of D among $\{S_i\}$.

• We have a characterisation of toric pairs and want a characterisation of pairs with a toric model.

Dim 2: Conjecture is a Theorem:
A CY pair (X, D) with maximal intersection has a toric model

Dim 3: Conjecture is false as $\exists$ maximal intersection CY pairs (X, D_X) with X non-rational.

Perhaps: (X, D_X) maximal intersection CY pair and X rational then (X, D_X) has a toric model.

§ Dimension 2 : (Blanc, Gross-Hacking-Keel, Hacking-Keating)

- (X,D) with X: smooth rational projective surface / $\mathbb{C}$
  $D \in |-K_X|$ a reduced nodal curve

  $D =$ irreducible nodal curve or a cycle of smooth rat$^{\underline{l}}$ curves.

rem : From [KX15] max intersection $\Rightarrow$ r.c $\Rightarrow$ rational

- A toric model of (X,D) is

$$\begin{tikzcd} & (\tilde{X},\tilde{D}) \arrow[dl] \arrow[dr] & \\ (X,D) & & (\bar{X},\bar{D}) \end{tikzcd}$$

blow up corners (sing points) of D and transforms

bu of valuations $a_E(K+D) = -1$

$\tilde{D} \to \bar{D}$: isomorphism
bu interior points of $\bar{D}$ (and proper transforms)

toric pair

bu of valuations $a_E(K+D) = 0$.

rem : $(X,D_X) \overset{\varphi}{\dashrightarrow} (Y,D_Y)$ crepant $\Rightarrow$ all basepoints of φ on D.

- **Prop** : 1) [GHK] Any CY surface (Y,D) with maximal intersection has a toric model.

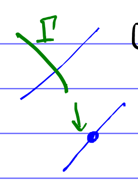
  2) [HK21 + Lutz] The toric model is unique up to well understood basic moves

Sketch proof : (1) Any CY surface (Y,D) with maximal intersection has a toric model.

Observe :

(i) if $p: Y \to Y'$ is the crepant blow down of a curve not contained in D and $D' = p_* D \subset Y'$ then
true for $(Y,D) \Leftrightarrow$ true for (Y',D')

Indeed:



$$\begin{tikzcd} & (\tilde{Y},\tilde{D}) \arrow[dl] \arrow[d] \arrow[drr] & & \\ (Y,D) \arrow[d] & (\tilde{Y}',\tilde{D}') \arrow[dl] \arrow[rr] & & (\bar{Y},\bar{D}) \\ (Y',D') & & & \end{tikzcd}$$

$\tilde{\Gamma}$

check that the image of $\tilde{\Gamma}$ is a point on $(\bar{Y},\bar{D})$

(ii) if $q: Y'' \to Y$ is the blow up of a node of D and if $D'' = q_*^{-1} D + E$
true for $(Y,D) \Leftrightarrow$ true for (Y'',D'')

$$\begin{tikzcd} (Y'',D'') \arrow[d] & (\tilde{Y}'',\tilde{D}'') \arrow[l] \arrow[d] \arrow[r] & (\bar{Y}'',D'') \arrow[d] \\ (Y,D) & (\tilde{Y},\tilde{D}) \arrow[l] \arrow[r] & (\bar{Y},\bar{D}) \end{tikzcd}$$

Using (i), (ii) wma Y = a minimal surface. $\nearrow \mathbb{P}^2$, $\searrow$ ruled $Y \to \mathbb{P}^1$

if $\mathbb{P}^2$, blowup a node of D to get to $Y = \mathbb{F}_1 \to \mathbb{P}^1$.

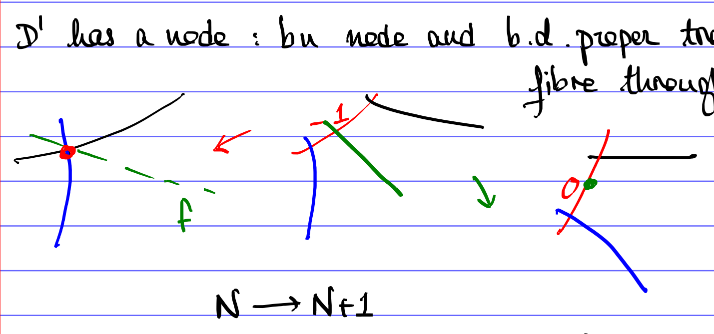
- Since $Y \to \mathbb{P}^1$ = a ruled surface $Y = \mathbb{F}_k$ some k.
- Let N = # components of D_Y that are fibres.

Recall: $\operatorname{Pic}(Y) \simeq \mathbb{Z}[\sigma] + \mathbb{Z}[f]$ $\quad \sigma^2 = -k$, $\sigma f = 1$, $f^2 = 0$
$-K_Y \sim 2\sigma + (k+2)f$

- See that $N \leq 2$ and if $N=2$ then D_Y has 4 components
  $D_Y = (f, \sigma + kf, f, \sigma)$ and (Y,D_Y) toric pair for suitable choice of toric structure on Y.

- Assume $N<2$ and let $D' \subset D_Y$ be the union of components not contained in fibres.

  if D' has a node : bu node and b.d. proper transform of fibre through node



$N \to N+1$

in this way we end up with
- $N=2$ done.
- $N=1$ $D = f \cup D'$
  and D' is a 2-section of $Y \to \mathbb{P}^1$
  $D' . \sigma = 1-k$

- if $\sigma \subset D'$: then $D = (\sigma, \sigma + (k+1)f, f)$ and since $\sigma . (\sigma + (k+1)f) = 1 \to$ go to $N=2$.
- wma $\sigma \not\subset D'$ and hence $k=0$ or 1.

<u>$k=0$</u> : $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{q'} \mathbb{P}^1$, $q \downarrow$ $\mathbb{P}^1$ $\quad$ D' and f are sections of the 2nd ruling.
bu nodes as before $\to N=2$

<u>$k=1$</u> : σ disjoint from D', bd σ to get to $\mathbb{P}^2$ and bu one of nodes of $D' \cup f$ $\Rightarrow$ bu a node and get back to $N=2$.

(2). The toric model is unique up to well understood basic moves

<u>Fact</u> : Any smooth proj toric surface is obtained from $\mathbb{P}^2$ by a sequence of toric blow ups and blow downs.

Assume $(\bar{Y}_1, \bar{D}_1)$ and $(\bar{Y}_2, \bar{D}_2)$ are crepant birational smooth proj. surfaces.

Then sequences of toric bu/bd such that

$$\underbrace{(\mathbb{P}^2, \Delta_1) \dashleftarrow (\bar{Y}_1, D_1) \overset{\varphi}{\dashrightarrow} (\bar{Y}_2, \bar{D}_2) \dashrightarrow (\mathbb{P}^2, \Delta_2)}_{F}$$

and consider the induced $(\mathbb{P}^2, \Delta_1) \overset{F}{\dashrightarrow} (\mathbb{P}^2, \Delta_2)$ crepant birational.

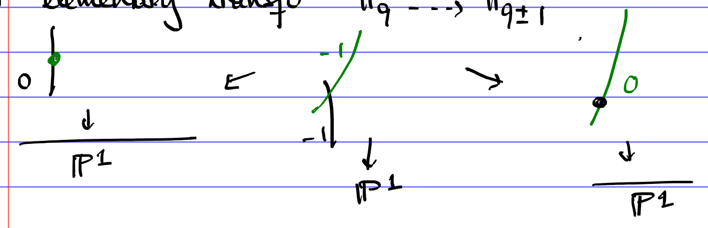
<u>Then</u> : [Blanc] $\exists$ a decomposition of F into crepant quadratic transformations (Noether-Castelnuovo type) $F = \phi_n \circ \dots \circ \phi_1$
st $\forall$ $1 \leq i \leq n$ the pair $(\mathbb{P}^2, (\phi_i \circ \dots \circ \phi_1)_* \Delta_1)$ is toric

[idea of proof : use NC while $(\mathbb{P}^2, (\phi_i \circ \dots \circ \phi_1)_* \Delta_1)$ remains toric, insert some additional quadratic transformations when not]

[Hacking-Keating] Use the version of the Sarkisov program for $\operatorname{Bir} \mathbb{P}^2$ as in [CKS]

Namely: Th: $\forall$ $\mathbb{P}^2 \overset{F}{\dashrightarrow} \mathbb{P}^2$ is a composition of the following elementary "links"

(1) $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\tau} \mathbb{P}^1 \times \mathbb{P}^1$ involution exchanging rulings
(2) $\varepsilon : \mathbb{F}_1 \to \mathbb{P}^2$ bu of a pt $\quad$ (3) $\mathbb{P}^2 \dashrightarrow \mathbb{F}_1$ inverse
(3) elementary transfo $\mathbb{F}_q \dashrightarrow \mathbb{F}_{q \pm 1}$



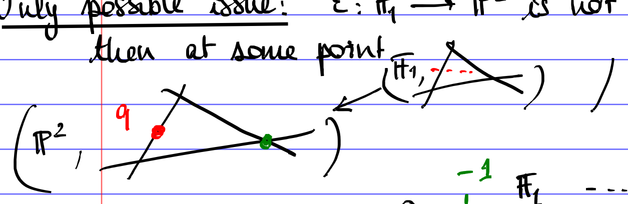
<u>rem</u> : Proof of theorem is inductive on Sarkisov deg.
(has to do with degree of linear system defining map to $\mathbb{P}^2$) and multiplicities.

[HK:] Any 2 toric models are related by a sequence of elementary transformations and toric blowups.

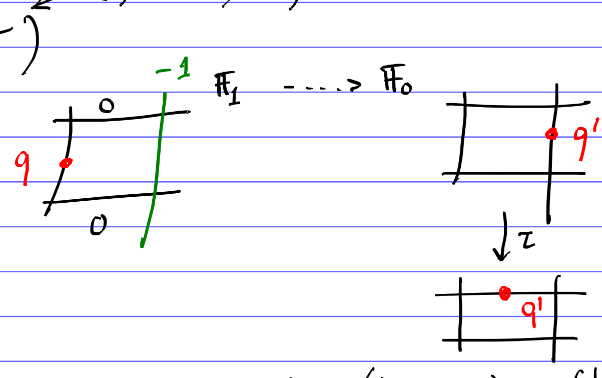
<u>Remark</u> : For elementary transformation. We have
$$\begin{tikzcd} & (\tilde{Y},\tilde{D}) \arrow[dl] \arrow[dr] & \\ (\bar{Y}_1,\bar{D}_1) \arrow[rr, dashed] & & (\bar{Y}_2,\bar{D}_2) \end{tikzcd}$$
and there are components of $\bar{D}_1$ Γ_i, Γ_j associated to rays $\mathbb{R}_{\geq 0} v$ and $\mathbb{R}_{\leq 0}(-v)$ defining $Y \to \mathbb{P}^1$
Fix Γ_i with m_i = number of int. points bu >0 then elementary transfo :
$m_i \to m_{i-1}$ $\quad$ $n_i = \Gamma_i^2 \to n_i - 1$
$m_j \to m_{j+1}$ $\quad$ $n_j = \Gamma_j^2 \to n_j + 1$

<u>rem</u> At each stage, non toric blow ups are at base points of the map F in the NC/CKS theorem.

<u>Only possible issue</u>: $\varepsilon : \mathbb{F}_1 \to \mathbb{P}^2$ is not necessarily a point b.u.
then at some point

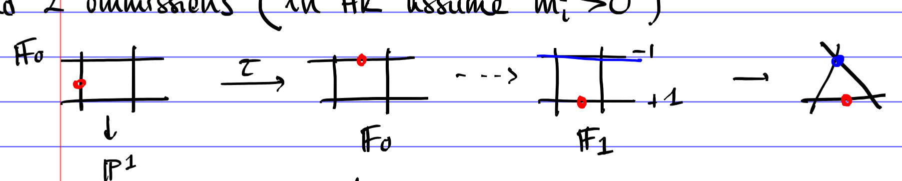


replace with:



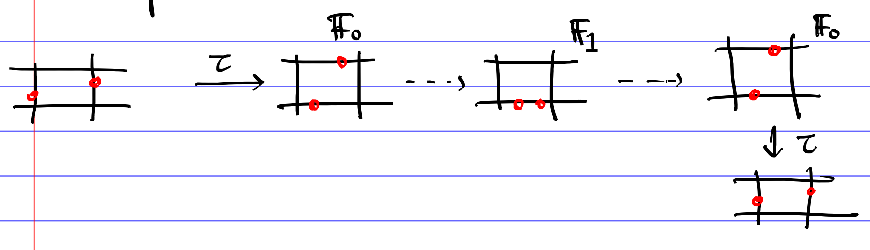
Then Sarkisov degree goes from $\left(\frac{d}{3}, m, n\right) \rightsquigarrow \left(\frac{d-n}{2}, \dots\right)$ $n > d/3$

And 2 ommissions (in HK assume $m_i > 0$)



one base point $x+y > b/2$.

or 2 base points . . .



§ Dimension 3.

Will see that (X, D_X) maximal intersection CY pair

$$\not\Downarrow$$

(X, D_X) has a toric model.

Key point: If (X, D_X) has a toric model, X is rational.

• **Recall**: Let X be a terminal $\mathbb{Q}$-factorial Fano 3-fold with $\rho = 1$. Then X is birationally rigid if $\forall\, Y \to T$ Mori fibre space with

$$\begin{array}{c} X \dashrightarrow Y \\ \phantom{X \dashrightarrow{}} \downarrow \\ \phantom{X \dashrightarrow{}} T \end{array}, \quad X \simeq Y.$$

Theorem [IM, Cheltsov, Mella] Let $X_4 \subseteq \mathbb{P}^4$ be a quartic hypersurface with no worse than ordinary double points. Then if $|\mathrm{Sing}(X)| \leq 8$ X is $\mathbb{Q}$ factorial and birationally rigid.

In practice — how does one prove a pair (X, D_X) has max. intersection?

• $\begin{array}{c}(\tilde{X}, D_{\tilde{X}}) \\ \downarrow \\ (X, D_X)\end{array}$ (t, dlt) modification

$\forall\, W$ stratum $\exists\, \mathrm{Diff}_W D_{\tilde{X}}$ s.t

$\left\{ \begin{array}{l} (W, \mathrm{Diff}_W D_{\tilde{X}}) \text{ is a lc CY pair} \\ K_W + \mathrm{Diff}_W D_{\tilde{X}} \sim_{\mathbb{Q}} K_{\tilde{X}} + D_{\tilde{X}}|_W \end{array} \right.$

Since $K_{\tilde{X}} + D_{\tilde{X}}$ Cartier
$D_{\tilde{X}}$ reduced $\Rightarrow \mathrm{Diff}_W D_{\tilde{X}} = \sum_{W \not\subset D_i} D_i$

So that: $\forall\, S$ irred component of $D_{\tilde{X}}$

link of $[S]$ in $\mathcal{D}(\tilde{X}, D_{\tilde{X}}) = \mathcal{D}(S, \mathrm{Diff}_S D_{\tilde{X}})$

$(\tilde{X}, D_{\tilde{X}})$ max intersect$^{\circ}$ $\Rightarrow (S, \mathrm{Diff}_S D_{\tilde{X}})$ max intersection
$\Downarrow$
slc

Ex 1: $X = \{x_1^4 + x_2^4 + x_3^4 + x_0 x_1 x_2 x_3 + x_4(x_0^3 + x_4^3) = 0\}$

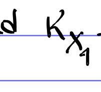
$D_X = X \cap \{x_4 = 0\}$

• $\mathrm{Sing}\, D_X = (1:0:0:0:0)$ and $\sim \underbrace{\{x^4 + y^4 + z^4 + xyz = 0\}}_{T_{444} \text{ cusp}}$

• D_X rational $\dashrightarrow \mathbb{P}^2_{x_1 x_2 x_3}$ defined outside 12 pts
$\{x_0 x_1 x_2 x_3 = x_1^4 + x_2^4 + x_3^4 = 0\}$

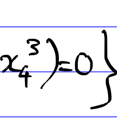
(X, D_X) not dlt.

$$\begin{array}{ccc} X_p & \longrightarrow & X \\ \cup & & \Psi \\ E & \longrightarrow & p \end{array}$$

$K_{X_p} + D_p + E = f^*(K_X + D)$
$(X_p, D_p + E)$ not dlt as
$D_p \cap E =$  $C_1 \cup C_2 \cup C_3$

$\{u = s_1 = s_4 = 0\} \cup \{u = s_2 = s_4 = 0\} \cup \{u = s_3 = s_4 = 0\}$

bu C_1 $E_1 = \mathbb{P}(\mathcal{N}_{C_1/X}) \simeq \mathbb{F}_2$ and $K_{X_1} + D_1 + E + E_1 = g_1^*(K_X + D)$

Now $(X_1, D_1 + E + E_1)$ dlt $\mathcal{D}(X_1, D_1 + E + E_1) =$ 
$(D_{|E_1} \cdot E_{|E_1} = 2)$

For simplicial $\mathcal{D}$: bu C_2.

Ex 2: $X = \{x_3(x_0^3 + x_1^3) + x_2^4 + x_0 x_1 x_2 x_3 + x_4(x_3^3 + x_4^3) = 0\}$

$D = \{x_4 = 0\} \cap X$ $\mathrm{Sing}\, X = 3$ odps
$\{x_2 = x_3 = x_4 = 0\} \cap \{x_0^3 + x_1^3 = 0\}$

$\mathrm{Sing}\, D_X = \mathrm{Sing}\, X \cup (0:0:0:1:0)$ T_{334} cusp.

$D_X \dashrightarrow \mathbb{P}^2$ projection from p. defined outside
$\{x_0^3 + x_1^3 + x_0 x_1 x_2 = x_0^3 + x_1^3 = 0\}$

$\tilde{X} \longrightarrow X$ bu at odp-s and p.
smooth
(t, dlt) modification? No!

(But in fact don't need to b-u. 3 odp-s.)

bu p: $\begin{array}{c}\tilde{X} \\ \cup \\ E\end{array} \longrightarrow X$ $D \cap E =$ curve, has a singular point

$\begin{array}{c}\tilde{\tilde{X}} \\ E, F\end{array} \longrightarrow \tilde{X} \longrightarrow X$ is a (t, dlt) mod and

$\mathcal{D}(\tilde{\tilde{X}}, D + E + F)$ is 2-dim.

Ex 3 (non normal boundary)

$$X = \{x_1^2 x_2^2 + x_1 x_2 x_3 l + x_3^2 q + x_4 f_3 = 0\}$$

l, q general linear / quadratic form in $x_0, \dots, x_3$
f_3 —— hgs form of deg 3 in $x_0, \dots, x_4$.

l, q, f_3 general $\Rightarrow$ X has 6 odps
$$\{x_1 = x_3 = x_4 = 0\} \cap \{f_3 = 0\} = L \cap \{f_3 = 0\}$$
$$\{x_2 = x_3 = x_4 = 0\} \cap \{f_3 = 0\} = L' \cap \{f_3 = 0\}.$$

D_X is non normal (xty 2 along L, L') and $p \in L \cap L'$

$p \in D_X \sim 0 \in (x^2y^2 + z^2 = 0)$ slc point
$\Rightarrow (X, D)$ has (t, lc) sing-s.

- Since $\mathrm{Sing}\, X \cap L \neq \emptyset$ $Bl_L X$ not $\mathbb{Q}$-factorial and consider

$$\begin{array}{ccc} f_L : X_L & \longrightarrow & X \\ \cup & & \cup \\ E_L & \longrightarrow & L \end{array}$$ divisorial extraction of L

f_L: bu 3 nodes, then bu proper transform of L then flop 3 lines $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ then contract 3 proper transforms of exc. divisors $\mathbb{P}^1 \times \mathbb{P}^1$ to $\frac{1}{2}(111)$

if then $\hat{X} \longrightarrow X_L \longrightarrow X$ $\quad (\hat{X}, D + E_L + E_{L'})$

$\uparrow$ divisorial extraction centered on L' $\quad$ | (t, dlt)